Section 1.4

Math 231

Hope College

- The cross product of two vectors $\vec{\mathbf{x}} = \langle x_1, x_2, x_3 \rangle$ and $\vec{\mathbf{y}} = \langle y_1, y_2, y_3 \rangle$ in $\mathbb{R}^3$ can be computed as

$$\begin{aligned} \vec{\mathbf{x}} \times \vec{\mathbf{y}} &= \det \begin{pmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \\ &= \vec{\mathbf{i}}(x_2 y_3 - x_3 y_2) - \vec{\mathbf{j}}(x_1 y_3 - x_3 y_1) + \vec{\mathbf{k}}(x_1 y_2 - x_2 y_1), \end{aligned}$$

  where $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$ and $\vec{\mathbf{k}}$ are the standard unit vectors in $\mathbb{R}^3$.

- Note that $\vec{\mathbf{x}} \times \vec{\mathbf{y}} = -\vec{\mathbf{y}} \times \vec{\mathbf{x}}$ for all $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^3$.
- **Theorem 1.35:** Let $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ be vectors in $\mathbb{R}^3$. Then
  1. $\vec{\mathbf{x}} \times \vec{\mathbf{y}}$ is orthogonal to both $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ with direction determined by the following right-hand rule: if you position your right hand so that your fingers curl in the direction from $\vec{\mathbf{x}}$ to $\vec{\mathbf{y}}$, then your thumb points in the appropriate direction for $\vec{\mathbf{x}} \times \vec{\mathbf{y}}$.
  2. $\vec{\mathbf{x}} \times \vec{\mathbf{y}}$ has length $\|\vec{\mathbf{x}}\| \, \|\vec{\mathbf{y}}\| \sin\theta$, where θ is the angle between $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$. (In other words, the length of $\vec{\mathbf{x}} \times \vec{\mathbf{y}}$ is equal to the area of the parallelogram P determined by $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$.)

Properties of Cross Products

- Note that $\vec{\mathbf{x}} \times \vec{\mathbf{y}} = -\vec{\mathbf{y}} \times \vec{\mathbf{x}}$ for all $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in \mathbb{R}^3$.
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Computing Volumes

Theorem 1.40: The volume of the parallelopiped T spanned by vectors $\vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}} \in \mathbb{R}^3$ is equal to $|(\vec{\mathbf{x}} \times \vec{\mathbf{y}}) \cdot \vec{\mathbf{z}}|$.